ON A CERTAIN FORM OF EXACT PARTICULAR SOLUTIONS OF A PLANE, VORTEX FREE GAS FLOW

(OB ODNOM VIDE TOCHNYKH CHASTNYKH RESHENII PLOSKOGO BEZVIKHREVOGO DVIZHENIIA GAZA)

PMM Vol. 24, No. 3, 1960, pp. 537-540

E. D. TOMILOV
(Tomsk)

(Received 20 July 1959)

1. Getting solutions in the hodograph plane. The flow function ψ of a plane steady-state vortex free motion of a nonviscous gas subject to the adiabatic law satisfies the equation of Chaplygin

$$\frac{\partial^2\psi}{\partial\sigma^2} + K\frac{\partial^2\psi}{\partial\theta^2} = 0 \qquad \left(\sigma = \int_{\tau}^{\tau_*} \frac{(1-\tau)^{\beta}}{2\tau}\, d\tau\right), \qquad K = \frac{1-(2\beta+1)\tau}{(1-\tau)^{2\beta+1}} \tag{1.1}$$

Here

$$\beta = \frac{1}{\varkappa - 1}, \qquad \tau = \frac{v^2}{v_m{}^2}, \qquad \tau_* = \frac{\varkappa - 1}{\varkappa + 1} = \frac{1}{2\beta + 1} \tag{1.2}$$

κ is the adiabatic exponent, v is the velocity, v_m is the limiting velocity in the gas, τ_* is the value of corresponding to the speed of sound, θ is the angle between the velocity and the x-axis, and K is the Chaplygin function. We shall seek a solution of Equation (1.1) in the form

$$\psi = P_0(\sigma) + \theta P_1(\sigma) + \ldots + \theta^{\nu} P_{\nu}(\sigma) + \ldots + \theta^{2n+1} P_{2n+1}(\sigma) \tag{1.3}$$

where the $P_{\nu}(\theta)$ are functions of σ only and n is an arbitrary non-negative integer.

For Equation (1.1) to hold the function P_{ν} must satisfy the following system of equations

$$P_{\nu}'' + (\nu+2)(\nu+1) K P_{\nu+2} = 0 \quad (\nu = 0, 1, \ldots, 2n-1), \qquad P_{2n}'' = 0, \qquad P_{2n+1}'' = 0 \tag{1.4}$$

The equations of this system yield two independent systems of equations for P_{ν} with even and odd indices. Integrating these equations we get

$$P_{2n-2k}=\frac{(-1)^k}{(2n-2k)!}\sum_{m=0}^{k}(a_{2m+1}G_{2k-2m}+b_{2m+1}F_{2k-2m}) \tag{1.5}$$

$$P_{2n-2k+1}=\frac{(-1)^k}{(2n-2k+1)!}\sum_{m=0}^{k}(a_{2m}G_{2k-2m}+b_{2m}F_{2k-2m}) \quad (k=0,1,\ldots,n) \tag{1.6}$$

where the functions F_{2i} and G_{2i} and the functions F_{2i-1} and G_{2i-1} ($i = 1, 2, \ldots, n$) which are indispensable in the sequel are given by the formulas

$$F_{2i-1}=\int_{\sigma_0}^{\sigma} KF_{2i-2}\,d\sigma, \quad F_{2i}=\int_{\sigma_0}^{\sigma} F_{2i-1}\,d\sigma=\int_{\sigma_0}^{\sigma}(\sigma-s)K(s)F_{2i-2}(s)\,ds \tag{1.7}$$

$$G_{2i-1}=\int_{\sigma_0}^{\sigma} KG_{2i-2}\,d\sigma, \quad G_{2i}=\int_{\sigma_0}^{\sigma} G_{2i-1}\,d\sigma=\int_{\sigma_0}^{\sigma}(\sigma-s)K(s)G_{2i-2}(s)\,ds \tag{1.8}$$

$$F_0=1, \qquad G_0=\sigma-\sigma_0 \tag{1.9}$$

a_i and b_i are constants of integration. σ_0 is the value of σ corresponding to an arbitrary value of $\tau = \tau_0$ in (1.1). From Chaplygin's equations

$$\frac{\partial\varphi}{\partial\sigma}=K\frac{\partial\psi}{\partial\theta}, \qquad \frac{\partial\varphi}{\partial\sigma}=-\frac{\partial\psi}{\partial\sigma} \tag{1.10}$$

we find an expression for the velocity potential ϕ in the form

$$\varphi=\sum_{\nu=0}^{2n+2}\theta^{\nu}Q_{\nu}(\sigma) \tag{1.11}$$

with

$$Q_{2n-2k}=\frac{(-1)^k}{(2n-2k)!}\sum_{m=0}^{k+1}(a_{2m}G_{2k-2m+1}+b_{2m}F_{2k-2m+1}) \quad (k=-1,0,1,\ldots,n) \tag{1.12}$$

$$Q_{2n-2k+1}=\frac{(-1)^{k+1}}{(2n-2k+1)!}\sum_{m=0}^{k}(a_{2m+1}G_{2k-2m+1}+b_{2m+1}F_{2k-2m+1}) \quad (k=0,1,2,\ldots,n) \tag{1.13}$$

Here

$$F_{-1}=0, \qquad G_{-1}=1 \tag{1.14}$$

It might seem that this solution is not new and that it could be obtained by the well-known method of Bergman when one takes as the "generating" harmonic function a sum of homogeneous harmonic polynomials. However, this is not so for the Bergman method would yield an infinite rather than a finite series in θ as the exact solution.

2. Some properties of the flows corresponding to the obtained solutions. By assigning different values to n and to the integration constants we obtain arbitrarily many different solutions. Since the System (1.4) splits into two independent systems, the Solution (1.3) may be regarded as the sum of two solutions, one with even indices and the other with odd indices:

$$\psi = \sum_{r=0}^{n} \theta^{2r} P_{2r}(\sigma), \qquad \psi = \sum_{r=0}^{n} \theta^{2r+1} P_{2r+1}(\sigma) \tag{2.1}$$

If a_{2n+1} is the only non zero constant in (1.5) and (1.13), then there remain only the functions $P_0 = (-1)^n a_{2n+1}$, $G_0 = aG_0$, $Q_1 = (-1)^{n+1} a_{2n+1} = -a$. In this case the first solution in (2.1) and Equation (1.11) yield $\psi = aG_0$, $\phi = -a\theta$, i.e. the gas flow is of the vortex point type.

In the sequel we assume everywhere $\sigma_0 = 0$, the value corresponding to critical velocity. It is clear that this assumption is no restriction of generality and is convenient for the study of sonic motions since in the case of a sonic flow the functions F and G with different indices other than $F_0 = G_{-1} = 1$ vanish.

If we then put all the b_i in (1.5) and (1.6) with the possible exception of b_{2n+1} equal to zero then all the functions P_ν other than $P_0 = (-1)^n b_{2n+1}$ vanish on the Mach line. Hence $\psi =$ const on the Mach line separating the subsonic and supersonic flows and this line is a flow line. Thus in this particular case the flow has the same property as a flow due to an isolated vortex.

Similarly, if in (1.6) and (1.12) we put all the coefficients other than b_{2n} equal to zero, then the only non zero functions are $P_1 = (-1)^n b_{2n} = b$, $Q_0 = (-1)^n b_{2n} F_1 = bF_1$. In this case the second solution in (2.1) and Equation (1.11) give $\psi = b\theta$, $\phi = bF_1$, i.e. a solution for a source in the gas.

It is well known that in case of a shock free transition through the Mach line it is necessary that $\partial\psi/\partial\theta = 0$ on that line. (1.3), (1.5) and (1.6) imply that on the Mach line

$$\frac{\partial\psi}{\partial\sigma} = \sum_{k=0}^{n} (-1)^k \left\{\frac{a_{2k}}{(2n-2k+1)!}\,\theta^{2n-2k+1} + \frac{a_{2k+1}}{(2n-2k)!}\,\theta^{2n-2k}\right\} \tag{2.2}$$

It follows that the transition of the gas through the Mach line is possible only for those sections for which not all of the coefficients vanish simultaneously. In the opposite case continuation of a vortex free flow beyond the Mach line is impossible. This agrees with the well-known property of a gas flow of the source type.

3. Computations. The functions F and G are not subject to any specific conditions and can be computed in advance for all problems.

In the sequel we put $\kappa = 1.4$ so that $\beta = 2.5$. Then by (1.1)

$$K=\frac{1-6\tau}{(1-\tau)^6} \tag{3.1}$$

The substitution $\tau = 1 - z^2$ reduces the expression for σ in (1.1) to the integrable form

$$\sigma=\int\limits_{z_*}^{z}\frac{z^6\,dz}{1-z^2} \qquad \left(z_*=\sqrt{1-\tau_*}=\sqrt{\frac{2}{\varkappa+1}}=\sqrt{\frac{5}{6}}=0.91287092917\right) \tag{3.2}$$

Hence

$$\sigma=\frac{1}{2}\ln\frac{1+z}{1-z}-z-\frac{z^3}{3}-\frac{z^5}{5}-A \qquad (A=0.2512511362) \tag{3.3}$$

where A is the value of the variable terms for $z = z_*$.

The functions F_1 and F_2 can be obtained by direct computation of integrals. On the basis of (3.1) and (3.2) we get

$$F_1=\frac{1}{z^5}-\frac{1}{3z^3}-\frac{1}{z}+\frac{1}{2}\ln\frac{1+z}{1-z}-B \qquad (B=1.5883027569) \tag{3.4}$$

$$F_2=\frac{1}{5}\left[z^4+2z^2-3\ln(1-z^2)\right]+\frac{1}{2}\left(\frac{1}{4}\ln\frac{1+z}{1-z}-z-\frac{z^3}{3}-\frac{z^5}{5}-B\right)\ln\frac{1+z}{1-z}+$$
$$+B\left(z+\frac{z^3}{3}+\frac{z^5}{5}\right)-C \qquad (C=0.3435517463) \tag{3.5}$$

Integration by parts enables us to express all the functions G_{2i-1} in terms of the other functions, namely,

$$G_{2i-1}=\sum_{k=1}^{2i}(-1)^{k-1}F_kG_{2i-1-k} \qquad (i=1,\ldots,n+1) \tag{3.6}$$

Thus we need only determine all the other functions F and the functions G with even indices. To tabulate the values of these functions one can use formulas for approximate evaluation of integrals or other methods (e.g. the Taylor expansion about $\sigma = 0$).

4. Transition to hodograph plane. Substitution of (1.3) and (1.11) in the Chaplygin formulas

$$dx=\frac{1}{v_m\sqrt{\tau}}\left[\cos\theta\,d\varphi-\frac{\sin\theta}{(1-\tau)^{5/2}}\,d\psi\right],\qquad dy=\frac{1}{v_m\sqrt{\tau}}\left[\sin\theta\,d\varphi+\frac{\cos\theta}{(1-\tau)^{5/2}}\,d\psi\right] \tag{4.1}$$

and integration yeild the following expressions for the coordinates of the hodograph plane

$$x=-\frac{1}{v_m\sqrt{\tau}}(R\sin\theta+S\cos\theta)+C_1,\qquad y=-\frac{1}{v_m\sqrt{\tau}}(S\sin\theta-R\cos\theta)+C_2 \tag{4.2}$$

Here C_1 and C_2 are integration constants and R and S are functions of

σ and θ defined by the formulas

$$R = \sum_{k=0}^{2n+1} (a_k e_k + b_k f_k), \qquad S = \sum_{k=0}^{2n+1} (a_k g_k + b_k h_k) \tag{4.3}$$

where a_k and b_k are the same arbitrary constants as before and the functions e_k and g_k have the form (4.4) and (4.5) below (depending on whether $k = 2m$ or $k = 2m + 1$)

$$\begin{aligned} e_{2m} &= \sum_{r=0}^{n-m} \frac{(-1)^{n-r}}{(2r+1)!} p_{2r+1} \{G_{2(n-m-r)-1} - qG_{2(n-m-r-1)}\} \\ e_{2m+1} &= \sum_{r=0}^{n-m} \frac{(-1)^{n-r}}{(2r)!} p_{2r} \{G_{2(n-m-r)-1} - qG_{2(n-m-r-1)}\} \end{aligned} \tag{4.4}$$

$$\begin{aligned} g_{2m} &= \sum_{r=0}^{n-m} \frac{(-1)^{n-r}}{(2r)!} p_{2r} \{G_{2(n-m-r)-1} - qG_{2(n-m-r)}\} \\ g_{2m+1} &= \sum_{r=0}^{n-m} \frac{(-1)^{n-r-1}}{(2r+1)!} p_{2r+1} \{G_{2(n-m-r)-3} - qG_{2(n-m-r-1)}\} \end{aligned} \tag{4.5}$$

The functions f_k are obtained from the expression for the e_k by replacing in the latter the functions G by the functions F with the same index. In the same way one gets the functions h_k from the expressions for the g_k. Also

$$q = (1 - \tau)^{-5/2} \tag{4.6}$$

$$\begin{aligned} p_{2r} &= \theta^{2r} - 2r(2r-1)\theta^{2r-2} + \cdots + (-1)^{r-1} 2r(2r-1)\cdots 4.3.\theta^2 + (-1)^r (2r)! \\ p_{2r+1} &= \theta^{2r+1} - (2r+1)2r\theta^{2r-1} + \cdots + (-1)^r (2r+1)!\,\theta \qquad (r = 0, 1, \ldots, n) \end{aligned} \tag{4.7}$$

Finally, in (4.4) and (4.5) it is necessary to put $F_{-1} = F_{-2} = F_{-3} = G_{-2} = G_{-3} = 0$.

5. Examples. We consider gas flows corresponding to solutions of the first type (2.1) for $n = 1$ and $a_k = 0$ and of the second type (2.1) for $n = 0$ and $b_k = 0$. In the first case the dimensionless flow function ψ^0 takes the form

$$\psi^\circ = \theta^2 - 2F_2 \qquad \left(\psi^\circ = \frac{2\psi}{b_1}\right) \tag{5.1}$$

and in the second case the form

$$\psi^\circ = \theta\sigma \qquad \left(\psi^\circ = \frac{\psi}{a_0}\right) \tag{5.2}$$

To utilize (3.3), (3.4) and (3.5) for computations we tabulated the functions σ, F_1 and F_2 as functions of τ at intervals $\Delta\tau = 0.005$ for

$0 < r < 0.4$ and $\Delta r = 0.05$ for $r > 0.4$.

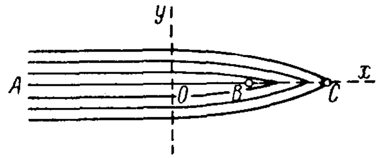

Fig. 1.

V. I. Mikuta and L. A. Donskaia participated in the computations and the determination of the form of the flows.

To obtain dimensionless coordinates we divided in the first case by b_1/v_m and in the second case by $-a_0/v_m$. It then turned out that the Mach line coincident with the limiting line had the form of an evolute of a circle of radius $q_*/\sqrt{r_*}$ starting at the point $x^0 = 0$, $y^0 = q_*/\sqrt{r_*}$ where q_* and r_* correspond to the critical velocity. The lines of flow issue from the points of the limiting line in a direction orthogonal to that line on its convex side and bend in the same direction as that line for subsonic velocities and, weakly, in the opposite direction for supersonic velocities. To the values $\theta > 0$ and $\theta < 0$ there correspond two limiting lines symmetric with respect to the y-axis. Only a flow associated with one of these limiting lines is possible since the lines of flow corresponding to two different evolutes intersect each other.

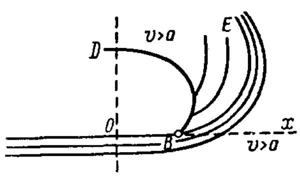

Fig. 2.

The flow corresponding to (5.2) turned out to be more interesting. Here one obtains a flow symmetric with respect to the x-axis whose qualitative features are represented in Fig. 1. At infinity to the left the velocities are zero. Then the flow gradually picks up speed and the flow lines converge very slowly and end on the segment BC of the x-axis. At B the velocity is equal to the velocity of sound and it decreases as we go from B to C. On AB $\psi^0 = 0$.

In this case the Mach line consists of two evolutes of a circle of radius $1/\sqrt{r_*}$ which originate at $x^0 = 1/\sqrt{r_*}$, $y^0 = 0$ (the point B in Fig. 1) and are branches of the line of flow $\psi^0 = 0$. The limiting line also issues from B in a direction perpendicular to the x-axis. In this

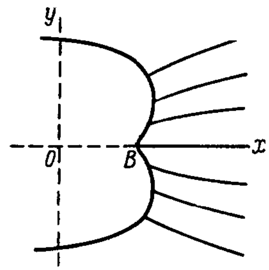

Fig. 3.

case an unsymmetric, mixed gas flow depicted in Fig. 2 is possible. Here ABE is a line of the flow $\psi^0 = 0$ which appears on the segment BE of the Mach line and BD is the limiting line. Below and to the right of ABE is the flow with subsonic velocity and the lines of flow gradually converge as a result of increased velocity along these lines. In the region DBE the flow has supersonic velocity. Reflection in the x-axis yields another such flow corresponding to negative values of θ.

Finally when we pass to the next sheet of the Riemann surface we obtain a pure supersonic flow whose qualitative features are represented in Fig. 3. Here the flow can be assumed to be symmetric. The part of the x-axis to the right of B is a line of the flow $\psi = 0$.

BIBLIOGRAPHY

1. Chaplygin, S.A., O gazovykh struiakh (On gas jets). *Sobr. soch., V.* II, Gostekhisdat, 1948.

2. Mises, E. and Shiffer, M., On Bergman's method for integrating the equations of a plane flow of a compressible fluid. Symposium article 'Problems of Mechanics' Pages 489-518, 1955.

Translated by